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Negative even grade mKdV hierarchy and its soliton solutions

J F Gomes, G Starvaggi França, G R de Melo and A H Zimerman

Instituto de Física Teórica-IFT/UNESP, Rua Dr. Bento Teobaldo Ferraz, 271, Bloco II,
01140-070, São Paulo, SP, Brazil

E-mail: jfg@ift.unesp.br, guisf@ift.unesp.br, gmelo@ift.unesp.br and zimerman@ift.unesp.br

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Abstract

In this paper, we provide an algebraic construction for the negative even mKdV hierarchy which gives rise to time evolutions associated with even graded Lie algebraic structures. We propose a modification of the dressing method, in order to incorporate a non-trivial vacuum configuration and construct a deformed vertex operator for $\hat{sl}(2)$ that enable us to obtain explicit and systematic solutions for the whole negative even grade equations.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The odd mKdV hierarchy consists of a series of nonlinear equations of motion associated with certain odd graded Lie algebraic structures such that each equation corresponds to a time evolution according to time $t = t_{2n+1}$ [1]. Since these are directly associated with odd graded operators, they are dubbed *odd order mKdV hierarchy*.

In this paper, we employ the algebraic technique which, for positive order, the graded structure of the zero curvature representation imposes severe restrictions so that only odd times are allowed. For negative order however, the structure is less restrictive and also gives rise to a subclass of equations of motion, described by time evolutions associated with *negative even grades*. These are constructed by the Lax operator in terms of an affine graded Lie algebra, $\hat{sl}(2)$, which from the zero curvature representation generates systematically a series of nonlinear integrable equations.

Recent interest in the negative flows within the KdV system linked with nonholonomic deformations has been considered in [2–4]. Moreover, mixed positive and negative hierarchies were also considered in [5–7].

By considering a special case of the Zakharov–Shabat AKNS spectral problem and using recursion techniques, a class of integrable equations was considered [8] in order to develop a negative order mKdV hierarchy as well as to obtain some parametric type of solutions.

Here, in our approach, the simplest case of the even mKdV where $t = t_{-2}$, is studied in detail and a crucial observation that a trivial zero constant solution is not admissible leads us to extend the dressing method to incorporate non-zero constant vacuum solutions. This implies in deforming the usual vertex operators in such a way that preserves the nilpotency property which is peculiar in solving soliton solutions. Employing the modified dressing formalism, we construct multi-soliton solutions for the whole negative even grade mKdV hierarchy.

In section 2, we discuss the algebraic formalism for positive and negative hierarchies [9, 10]. In particular, the construction of the equation of motion for $t = t_{-2}$. Such an equation agrees with that proposed in [8] using recursion operator techniques. In section 3, we discuss the dressing formalism [11–15] to construct soliton solutions for the odd hierarchy. In this case, the formalism is based upon a constant zero vacuum solution which, by gauge transformation, generates multi-soliton solutions. In section 4, we extend the dressing formalism to the negative even hierarchy, by introducing a non-zero constant vacuum configuration. We then construct the deformed vertex operators which generate explicitly the multi-soliton solutions. Some details involving the explicit calculation of matrix elements of products of vertex operators and the proof of their nilpotency are described in the appendix.

2. Positive and negative hierarchies

Consider the *positive mKdV hierarchy* associated with $\mathcal{G} = sl(2)$ with generators satisfying $[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$, $[E_{\alpha}, E_{-\alpha}] = h$ and grading operator $Q = 2\lambda \frac{d}{dx} + \frac{1}{2}h$. Q decomposes the affine Lie algebra $\hat{sl}(2)$ into graded subspaces, $\hat{\mathcal{G}} = \oplus_i \mathcal{G}_i$,

$$\mathcal{G}_{2m} = \{h^{(m)} = \lambda^m h\}, \quad \mathcal{G}_{2m+1} = \{\lambda^m (E_{\alpha} + \lambda E_{-\alpha}), \lambda^m (E_{\alpha} - \lambda E_{-\alpha})\} \quad (2.1)$$

$m = 0, \pm 1, \pm 2, \dots$. The zero curvature representation for the mKdV hierarchy reads

$$[\partial_x + E^{(1)} + A_0, \partial_{t_n} + D^{(n)} + D^{(n-1)} + \dots + D^{(0)}] = 0, \quad (2.2)$$

where $D^{(j)} \in \mathcal{G}_j$, $E^{(2n+1)} = \lambda^n (E_{\alpha} + \lambda E_{-\alpha})$ and $A_0 = vh$ contains the field variable $v = v(x, t_n)$. A more subtle structure arises if one considers the decomposition $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$ where $\mathcal{K} = \{\lambda^n (E_{\alpha} + \lambda E_{-\alpha})\}$ denotes the Kernel of $E \equiv E^{(1)}$, i.e., $\mathcal{K} = \{k \in \hat{\mathcal{G}} | [E, k] = 0\}$ and \mathcal{M} is its complement. We assume that E is semi-simple in the sense that this second decomposition is such that

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}. \quad (2.3)$$

Equation (2.2) can be decomposed grade by grade and solved for $D^{(j)}$. For instance, the highest grade in (2.2) yields

$$[E, D^{(n)}] = 0 \Rightarrow D^{(n)} = D_{\mathcal{K}}^{(n)} \in \mathcal{K}. \quad (2.4)$$

Since by (2.1) \mathcal{K} has grade $2m + 1$, this last equation implies that $n = 2m + 1$ and hence $t_n = t_{2m+1}$, showing that only *odd* grades are admissible for positive mKdV hierarchy (2.2). Moving down grade by grade and using the symmetric space structure (2.3), equation (2.2) allows one to solve for all $D^{(j)} = D_{\mathcal{K}}^{(j)} + D_{\mathcal{M}}^{(j)}$, $j = 0, \dots, n$. In particular, the zero grade projection in \mathcal{M} yields the equation of motion

$$\partial_{t_n} A_0 - \partial_x D_{\mathcal{M}}^{(0)} - [A_0, D_{\mathcal{K}}^{(0)}] = 0, \quad (2.5)$$

where we have taken into account that $A_0 \in \mathcal{M}$. Equation (2.5) represents a series of nonlinear evolution equations associated with time t_{2m+1} . Choosing $m = 1$ for example, we will obtain the well-known mKdV equation [1, 9].

The same, however, does not happen for the *negative mKdV hierarchy* [9, 10], i.e., for $n < 0$. Let us consider the zero curvature representation

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{-n}} + D^{(-n)} + D^{(-n+1)} + \dots + D^{(-1)}] = 0. \quad (2.6)$$

Here, the lowest grade projection,

$$\partial_x D^{(-n)} + [A_0, D^{(-n)}] = 0, \quad (2.7)$$

yields a nonlocal equation for $D^{(-n)}$. The second lowest projection of grade $-n + 1$ leads to

$$\partial_x D^{(-n+1)} + [A_0, D^{(-n+1)}] + [E^{(1)}, D^{(-n)}] = 0 \quad (2.8)$$

which determines $D^{(-n+1)}$. The same mechanism works recursively until we reach the zero grade equation

$$\partial_{t_{-n}} A_0 + [E^{(1)}, D^{(-1)}] = 0 \quad (2.9)$$

which gives the time evolution for the field in A_0 according to time t_{-n} . The simplest model of this sub-hierarchy is obtained for $n = 1$, for which the following equations arrive from the zero curvature (2.6),

$$\partial_x D^{(-1)} + [A_0, D^{(-1)}] = 0, \quad \partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] = 0. \quad (2.10)$$

These equations can be solved in general if we parametrize the fields as

$$D^{(-1)} = B^{-1} E^{(-1)} B, \quad A_0 = B^{-1} \partial_x B, \quad B = \exp(\mathcal{G}_0) \quad (2.11)$$

in terms of the zero grade subalgebra \mathcal{G}_0 . Spacetime is associated with the light-cone coordinates \bar{z}, z as $x = \bar{z}, t_{-1} = z$. The time evolution is then given by the Leznov–Saveliev equation,

$$\partial_{t_{-1}} (B^{-1} \partial_x B) = [E^{(1)}, B^{-1} E^{(-1)} B] \quad (2.12)$$

which for $\hat{sl}(2)$ with principal gradation $Q = 2\lambda \frac{d}{dx} + \frac{1}{2}h$, yields the sinh-Gordon equation [13]

$$\partial_{t_{-1}} \partial_x \phi = e^{2\phi} - e^{-2\phi}, \quad B = e^{\phi h}. \quad (2.13)$$

Note that from the definition of A_0 and the parametrization (2.11), we find the following relation between ϕ and v : $A_0 = v h = B^{-1} \partial_x B \Rightarrow v = \partial_x \phi$.

We now propose the first nontrivial example for the *negative even* sub-hierarchy:

$$\partial_x D^{(-2)} + [A_0, D^{(-2)}] = 0, \quad (2.14)$$

$$\partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] = 0, \quad (2.15)$$

$$\partial_{t_{-2}} A_0 - [E^{(1)}, D^{(-1)}] = 0. \quad (2.16)$$

From grading (2.1) and decomposition (2.3) we find

$$D^{(-2)} = c_{-2} \lambda^{-1} h, \quad D^{(-1)} = a_{-1} (\lambda^{-1} E_\alpha + E_{-\alpha}) + b_{-1} (\lambda^{-1} E_\alpha - E_{-\alpha}). \quad (2.17)$$

From equations (2.14) and (2.15) we find, $c_{-2} = \text{const}$ and

$$\partial_x (a_{-1} + b_{-1}) + 2v(a_{-1} + b_{-1}) - 2c_{-2} = 0, \quad (2.18)$$

$$\partial_x (a_{-1} - b_{-1}) - 2v(a_{-1} - b_{-1}) + 2c_{-2} = 0,$$

which are ordinary differential equations with solution

$$\begin{aligned} a_{-1} + b_{-1} &= 2c_{-2} \exp(-2d^{-1}v)d^{-1}(\exp(2d^{-1}v)), \\ a_{-1} - b_{-1} &= -2c_{-2} \exp(2d^{-1}v)d^{-1}(\exp(-2d^{-1}v)). \end{aligned} \quad (2.19)$$

In (2.19), we have denoted $d^{-1}f = \int^x f(x') dx'$. Having determined $D^{(-1)}$, the evolution equation associated with time t_{-2} is then given by equation (2.16):

$$\partial_{t_{-2}} v + 2c_{-2} e^{-2d^{-1}v} d^{-1}(e^{2d^{-1}v}) + 2c_{-2} e^{2d^{-1}v} d^{-1}(e^{-2d^{-1}v}) = 0. \quad (2.20)$$

Note that $v = 0$ is not a solution to (2.20) for $c_{-2} \neq 0$. Differentiating twice with respect to x , and setting $c_{-2} = 1$ for convenience, we find the local equation

$$v_{xx t_{-2}} - 4v^2 v_{t_{-2}} - \frac{v_x v_{x t_{-2}}}{v} - 4 \frac{v_x}{v} = 0. \quad (2.21)$$

Equation (2.21) is already obtained in [8] using recursion operator techniques.

3. Odd hierarchy solutions

In order to employ the dressing method to construct soliton solutions, we now introduce the full $\hat{sl}(2)$ affine Kac–Moody algebra with central extensions (central extensions are necessary in order to ensure highest weight representations of the affine algebra, see for instance [12, 13, 15]):

$$\begin{aligned} [h^{(m)}, h^{(n)}] &= 2m\delta_{m+n,0}\hat{c}, \\ [h^{(m)}, E_{\pm\alpha}^{(n)}] &= \pm 2E_{\pm\alpha}^{(m+n)}, \\ [E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}] &= h^{(m+n)} + m\delta_{m+n,0}\hat{c} \end{aligned} \quad (3.1)$$

together with the derivation operator \hat{d} such that,

$$[\hat{d}, T_a^{(n)}] = nT_a^{(n)}, \quad T_a^{(n)} = \{h^{(n)}, E_{\pm\alpha}^{(n)}\}. \quad (3.2)$$

The grading operator now reads $Q = 2\hat{d} + 1/2h^{(0)}$. A well-established method for determining soliton solutions is to choose a vacuum solution and then to map it into a non-trivial solution by gauge transformation (dressing) [12, 13]. The zero curvature condition (2.2) or (2.6) implies pure gauge connections, $A_x = T^{-1}\partial_x T = E + A_0$ and $A_{t_n} = T^{-1}\partial_{t_n} T = D^{(n)} + \dots + D^{(0)}$ or $A_{t_{-n}} = T^{-1}\partial_{t_{-n}} T = D^{(-n)} + \dots + D^{(-1)}$, respectively. Suppose there exists a vacuum solution satisfying

$$A_{x,\text{vac}} = E^{(1)} - t_k \delta_{k+1,0}\hat{c}, \quad A_{t_k,\text{vac}} = E^{(k)}, \quad (3.3)$$

where the central term contribution in $A_{x,\text{vac}}$ is a consequence of nonzero commutator, $[E^{(k)}, E^{(l)}] = \frac{1}{2}(k-l)\delta_{k+l,0}\hat{c}$, for (k, l) odd integers (see [12, 13, 15]).

The solution for $A_{x,\text{vac}} = T_0^{-1}\partial_x T_0$ and $A_{t_k,\text{vac}} = T_0^{-1}\partial_{t_k} T_0$ is therefore given by

$$T_0 = \exp(xE^{(1)}) \exp(t_k E^{(k)}). \quad (3.4)$$

The dressing method is based on the assumption of the existence of two gauge transformations, generated by Θ_{\pm} , mapping the vacuum into non-trivial configuration, i.e.

$$A_x = (\Theta_{\pm})^{-1} A_{x,\text{vac}} \Theta_{\pm} + (\Theta_{\pm})^{-1} \partial_x \Theta_{\pm}, \quad (3.5)$$

$$A_{t_k} = (\Theta_{\pm})^{-1} A_{t_k,\text{vac}} \Theta_{\pm} + (\Theta_{\pm})^{-1} \partial_{t_k} \Theta_{\pm}. \quad (3.6)$$

where Θ_{\pm} are group elements of the form

$$\Theta_{-}^{-1} = e^{p^{(-1)}} e^{p^{(-2)}} \dots, \quad \Theta_{+}^{-1} = e^{q^{(0)}} e^{q^{(1)}} e^{q^{(2)}} \dots, \quad (3.7)$$

$p^{(-i)}$ and $q^{(i)}$ are linear combinations of grade $(-i)$ and (i) generators, respectively. As a consequence we relate

$$\Theta_{-}\Theta_{+}^{-1} = T_0^{-1} g T_0 \quad (3.8)$$

where g is an arbitrary constant group element.

By considering Θ_+ , the zero grade component of (3.5) admits solution

$$e^{q(0)} = B^{-1} e^{-v\hat{c}}, \tag{3.9}$$

where we have used $A_x = E^{(1)} + B^{-1}\partial_x B + \partial_x v\hat{c} - t_k\delta_{k+1,0}\hat{c}$. From equation (3.8) we find

$$\dots e^{-p(-2)} e^{-p(-1)} B^{-1} e^{-v\hat{c}} e^{q(1)} e^{q(2)} \dots = T_0^{-1} g T_0 \tag{3.10}$$

hence

$$\langle \lambda' | B^{-1} | \lambda \rangle e^{-v} = \langle \lambda' | T_0^{-1} g T_0 | \lambda \rangle, \tag{3.11}$$

where $|\lambda\rangle$ and $\langle \lambda'|$ are annihilated by $\mathcal{G}_>$ and $\mathcal{G}_<$, respectively. Explicit space time dependence for the field in \mathcal{G}_0 , defined in (2.13), is given by choosing specific matrix elements (A.2):

$$e^{-v} = \langle \lambda_0 | T_0^{-1} g T_0 | \lambda_0 \rangle, \quad e^{-\phi-v} = \langle \lambda_1 | T_0^{-1} g T_0 | \lambda_1 \rangle, \tag{3.12}$$

where $|\lambda_i\rangle, i = 0, 1$ correspond to highest weight states, i.e. annihilated by positive grade operators. Suppose we now write the constant group element g as

$$g = \exp\{F(\gamma)\}, \tag{3.13}$$

where γ is a complex parameter and we choose $F(\gamma)$ to be an eigenstate of $E^{(k)}$, i.e.

$$[E^{(k)}, F(\gamma)] = f^{(k)}(\gamma)F(\gamma), \tag{3.14}$$

where $f^{(k)}$ are specific functions of γ . It therefore follows that

$$T_0^{-1} g T_0 = \exp\{\rho(\gamma)F(\gamma)\}, \tag{3.15}$$

where

$$\rho(\gamma) = \exp\{-t_k f^{(k)}(\gamma) - x f^{(1)}(\gamma)\}. \tag{3.16}$$

For more general cases in which

$$g = \exp\{F_1(\gamma_1)\} \exp\{F_2(\gamma_2)\} \dots \exp\{F_N(\gamma_N)\} \tag{3.17}$$

with

$$[E^{(k)}, F_i(\gamma_i)] = f_i^{(k)}(\gamma_i)F_i(\gamma_i) \tag{3.18}$$

we find

$$T_0^{-1} g T_0 = \exp\{\rho_1(\gamma_1)F_1(\gamma_1)\} \exp\{\rho_2(\gamma_2)F_2(\gamma_2)\} \dots \exp\{\rho_N(\gamma_N)F_N(\gamma_N)\}, \tag{3.19}$$

where

$$\rho_i(\gamma_i) = \exp\{-t_k f_i^{(k)}(\gamma_i) - x f_i^{(1)}(\gamma_i)\}. \tag{3.20}$$

The specific eigenstate, in this case of $\hat{sl}(2)$, is given by

$$F(\gamma) = \sum_{n=-\infty}^{\infty} \left(h^{(n)} - \frac{1}{2}\delta_{n,0}\hat{c} \right) \gamma^{-2n} + (E_{\alpha}^{(n)} - E_{-\alpha}^{(n+1)}) \gamma^{-2n-1} \tag{3.21}$$

whose eigenvalues are obtained from

$$[E^{(k)}, F(\gamma)] = -2\gamma^k F(\gamma). \tag{3.22}$$

From equations (3.12) we obtain solutions for ϕ (or equivalently $v = \partial_x \phi$) for the whole odd hierarchy, i.e., for all variables t_k in equation (3.20). Observe that in equations (3.14) and (3.18), $F(\gamma)$ is a simultaneous eigenstate of both $E^{(1)}$ and $E^{(k)}$ and belong to the kernel \mathcal{K} . The above argument is therefore valid only for $k = 2m + 1, m = 0, \pm 1, \pm 2, \dots$ since \mathcal{K} contains only odd grade elements. Then, this method gives explicit solutions for both, equations (2.2) and (2.6) for $n = k = 2m + 1$. See, for instance [13–15].

4. Negative even hierarchy solutions

In order to modify the dressing method to construct systematic solutions of equations like (2.20) or (2.21), we note that $v = 0$ cannot be the solution to (2.21). Therefore, let us propose the simplest vacuum configuration

$$\begin{aligned} A_{x,\text{vac}} &= (E_\alpha^{(0)} + E_{-\alpha}^{(1)}) + v_0 h^{(0)} - \frac{1}{v_0} t_{-2m} \delta_{m-1,0} \hat{c}, \\ A_{t_{-2m},\text{vac}} &= \frac{1}{v_0} (E_\alpha^{(-m)} + E_{-\alpha}^{(1-m)}) + h^{(-m)} \end{aligned} \quad (4.1)$$

with $v_0 = \text{const} \neq 0$. It is straightforward to verify the zero curvature equation

$$[\partial_x + A_{x,\text{vac}}, \partial_{t_{-2m}} + A_{t_{-2m},\text{vac}}] = 0. \quad (4.2)$$

This *non-trivial vacuum* leads to the following modification of equation (3.4), but now for negative even grades,

$$T_0 = \exp \left\{ x (E_\alpha^{(0)} + E_{-\alpha}^{(1)} + v_0 h^{(0)}) \right\} \exp \left\{ \frac{t_{-2m}}{v_0} (E_\alpha^{(-m)} + E_{-\alpha}^{(1-m)} + v_0 h^{(-m)}) \right\}. \quad (4.3)$$

The analogous of equation (3.9) leads to

$$e^{q(0)} = B^{-1} e^{x v_0 h^{(0)}} e^{-v \hat{c}}. \quad (4.4)$$

Observe that consistency of the zero curvature representation with non-trivial vacuum configuration requires terms with mixed gradation in constructing T_0 as in (4.3). The solution is then given by

$$e^{-v} = \langle \lambda_0 | T_0^{-1} g T_0 | \lambda_0 \rangle \equiv \tau^+, \quad e^{-\phi + x v_0 - v} = \langle \lambda_1 | T_0^{-1} g T_0 | \lambda_1 \rangle \equiv \tau^- \quad (4.5)$$

and hence,

$$v = v_0 - \partial_x \ln \left(\frac{\tau^+}{\tau^-} \right), \quad v = \partial_x \phi. \quad (4.6)$$

In order to construct explicit soliton solutions we need a simultaneous eigenstate of $b_1 \equiv E_\alpha^{(0)} + E_{-\alpha}^{(1)} + v_0 h^{(0)}$ and $b_{-2m} \equiv (E_\alpha^{(-m)} + E_{-\alpha}^{(-m+1)} + v_0 h^{(-m)})$. Let

$$F(\gamma, v_0) = \sum_{n=-\infty}^{\infty} (\gamma^2 - v_0^2)^{-n} \left[h^{(n)} + \frac{v_0 - \gamma}{2\gamma} \delta_{n,0} \hat{c} + E_\alpha^{(n)} (\gamma + v_0)^{-1} - E_{-\alpha}^{(n+1)} (\gamma - v_0)^{-1} \right] \quad (4.7)$$

be our deformed vertex operator. A direct calculation shows that

$$\begin{aligned} [b_1, F(\gamma, v_0)] &= -2\gamma F(\gamma, v_0), \\ [b_{-2m}, F(\gamma, v_0)] &= -2\gamma (\gamma^2 - v_0^2)^{-m} F(\gamma, v_0). \end{aligned} \quad (4.8)$$

Therefore from (3.16) we find

$$\rho(\gamma, v_0) = \exp \left\{ 2\gamma x + \frac{2\gamma t_{-2m}}{v_0 (\gamma^2 - v_0^2)^m} \right\}. \quad (4.9)$$

It only remains to calculate the matrix elements in equations (4.5). They are shown in the appendix. Note that because of the nilpotency property of the vertex operator between matrix elements, as discussed in the appendix, the exponential series in equation (4.5) truncates, e.g, if we take,

$$g = \exp\{F(\gamma, v_0)\} \quad (4.10)$$

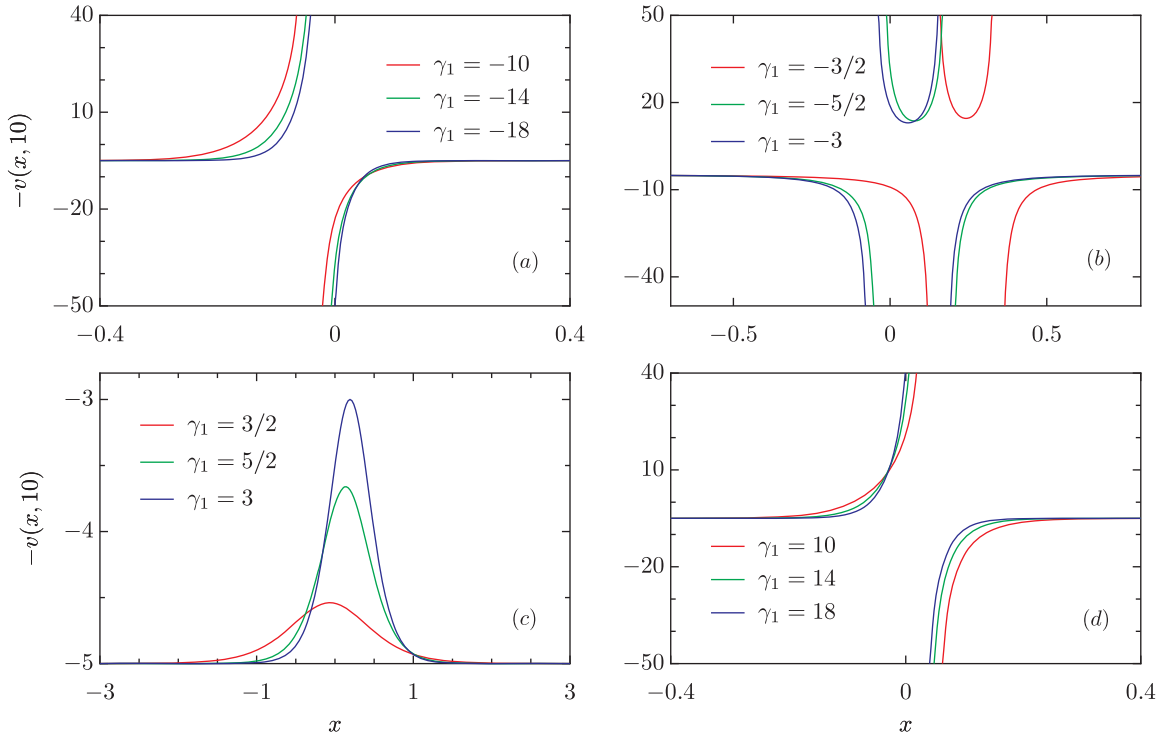


Figure 1. 1-soliton solutions for equation (2.21), $t_n = t_{-2}$ in mKdV hierarchy. In all these graphs we set $v_0 = 5$ and choose a fixed time $t_{-2} = 10$.

we have

$$\begin{aligned} \langle \lambda_a | T_0^{-1} g T_0 | \lambda_a \rangle &= \langle \lambda_a | \exp \{ \rho(\gamma, v_0) F(\gamma, v_0) \} | \lambda_a \rangle \\ &= 1 + \rho(\gamma, v_0) \langle \lambda_a | F(\gamma, v_0) | \lambda_a \rangle. \end{aligned} \tag{4.11}$$

Thus, from equation (4.6), we obtain *explicit* solutions for the *whole negative even hierarchy*. The introduction of a non-trivial vacuum configuration, v_0 , in the dressing method, seems to have the same effect as a change in the boundary conditions when looking for solutions of differential equations, as can be noted in equation (4.6). In this vein, we can say that our modified dressing approach implements a different boundary condition than the usual dressing of a trivial vacuum, largely used until now.

Let us introduce the shorthand notation:

$$c_i^\pm = \frac{v_0 \pm \gamma_i}{2\gamma_i}, \quad a_{ij} = \left(\frac{\gamma_i - \gamma_j}{\gamma_i + \gamma_j} \right)^2, \quad \rho_i = \exp \left\{ 2\gamma_i x + \frac{2\gamma_i t_{-2m}}{v_0(\gamma_i^2 - v_0^2)^m} \right\}. \tag{4.12}$$

The 1-soliton solution in equation (4.6) is obtained with one vertex as in (4.10). The explicit tau functions are

$$\tau^\pm = 1 + c_1^\pm \rho_1. \tag{4.13}$$

The 2-soliton solution is obtained with

$$g = \exp\{F(\gamma_1, v_0)\} \exp\{F(\gamma_2, v_0)\} \tag{4.14}$$

in (4.5), and then

$$\tau^\pm = 1 + c_1^\pm \rho_1 + c_2^\pm \rho_2 + c_1^\pm c_2^\pm a_{12} \rho_1 \rho_2. \tag{4.15}$$

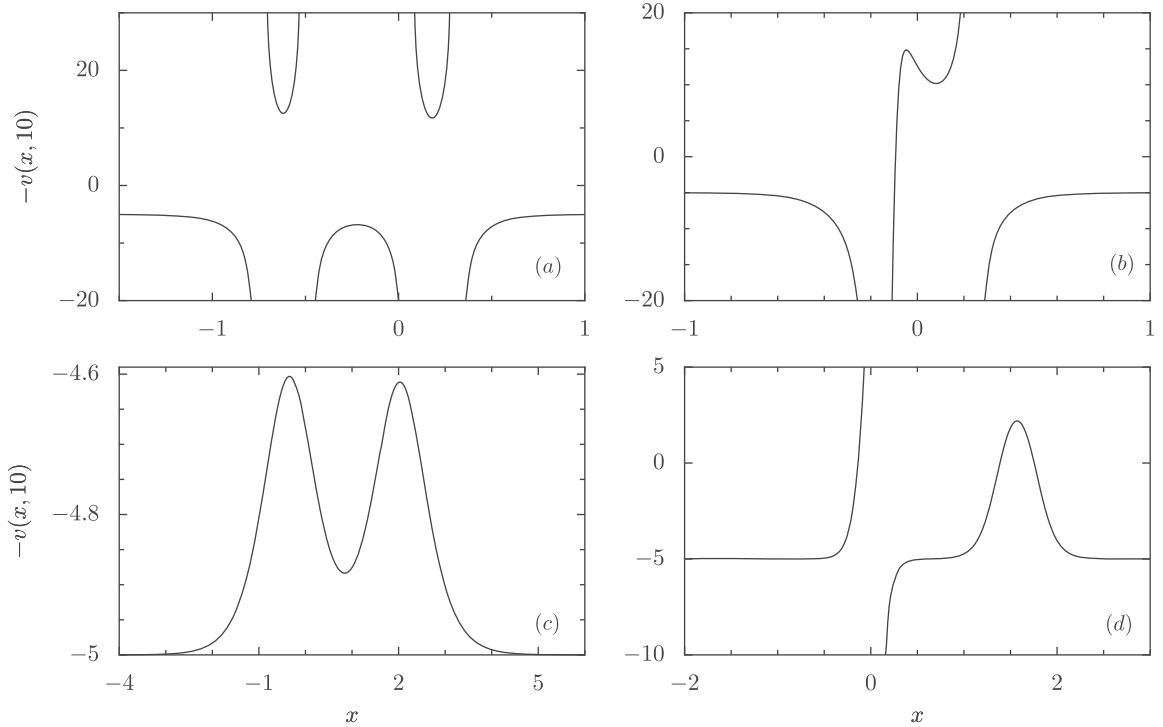


Figure 2. 2-soliton solutions for equation (2.21). $v_0 = 5$ and $t_{-2} = 10$. Parameters: (a) $\gamma_1 = -4, \gamma_2 = -3$; (b) $\gamma_1 = -4, \gamma_2 = -10$; (c) $\gamma_1 = 1.3, \gamma_2 = 1.5$; (d) $\gamma_1 = 4.8, \gamma_2 = 7$.

The 3-soliton solution, obtained as a product of three exponential vertices, is given by

$$\begin{aligned} \tau^\pm = & 1 + c_1^\pm \rho_1 + c_2^\pm \rho_2 + c_3^\pm \rho_3 + c_1^\pm c_2^\pm a_{12} \rho_1 \rho_2 + c_1^\pm c_3^\pm a_{13} \rho_1 \rho_3 + c_2^\pm c_3^\pm a_{23} \rho_2 \rho_3 \\ & + c_1^\pm c_2^\pm c_3^\pm a_{12} a_{13} a_{23} \rho_1 \rho_2 \rho_3. \end{aligned} \tag{4.16}$$

If we then substitute

$$g = \prod_{i=1}^n \exp\{F(\gamma_i, v_0)\} \tag{4.17}$$

in equations (4.5) we obtain the general n -soliton solution:

$$\tau^\pm = \sum_{J \subset I} \left(\prod_{i \in J} c_i^\pm \right) \left(\prod_{i,j \in J, i < j} a_{ij} \right) \prod_{i \in J} \rho_i \tag{4.18}$$

where $I = \{1, \dots, n\}$ and the sum is over all subsets J of I . These solutions present the same structure as those constructed from the trivial vacuum solution. They differ only by the deformation in (4.12) which now incorporates the parameter v_0 .

The solutions of equation (2.21), $t_{-2m} = t_{-2}$, are obtained by setting $m = 1$ in (4.12). Considering 1-soliton solution (4.13), we see that a critical behavior occurs when $\gamma_1 \rightarrow \pm v_0$ or $\gamma_1 \rightarrow 0$. So, we have *four* different regions to consider: $\gamma_1 < -v_0$; $-v_0 < \gamma_1 < 0$; $0 < \gamma_1 < v_0$; $\gamma_1 > v_0$. All these regions are considered separately in figure 1. These solutions keep their form for any t_{-2} . Note that three different types of behavior occur, figures 1(a) and (d) are of the same type, and have the same form as the usual *trivial vacuum* solutions of *odd grade* mKdV hierarchy, except from the fact that the solution is displaced by v_0 in the y -axis.

Figures 1(b) and (c) are different ones, and their form are not obtained from the trivial vacuum configuration. Also, note that $v \rightarrow v_0$ when $x \rightarrow \pm\infty$.

In figure 2, we show the 2-soliton solution for equation (2.21), where we illustrate the mixing of different types of solutions. Again, a different behavior emerges compared with the trivial vacuum 2-soliton solutions.

5. Conclusions

We extended the mKdV hierarchy to include *negative even grade* equations, based on a graded infinite-dimensional Lie algebra $\hat{sl}(2)$. This procedure systematically leads us to obtain new nonlinear integrable equations, e.g. equation (2.21) which was previously obtained in [8]. Our method can also provide other higher order integro-differential equations, like for example

$$\begin{aligned} \partial_x \partial_{t-4} \phi = & 4e^{-2\phi} d^{-1} [e^{2\phi} d^{-1} (e^{-2\phi} d^{-1} e^{2\phi} + e^{2\phi} d^{-1} e^{-2\phi})] \\ & - 4e^{2\phi} d^{-1} [e^{-2\phi} d^{-1} (e^{-2\phi} d^{-1} e^{2\phi} + e^{2\phi} d^{-1} e^{-2\phi})]. \end{aligned} \tag{5.1}$$

This subhierarchy of even grade equations is not solved by the usual dressing method, based on a trivial vacuum configuration. Nevertheless, we also extended the dressing method to incorporate a constant non-trivial vacuum configuration v_0 .

Remarkably, all these modifications lead us to obtain solutions for the whole negative even grade mKdV subhierarchy, in particular for equation (2.21). Our solutions for equation (2.21) do not appear in [8]. The introduction of the constant vacuum, v_0 , showed that the simplest 1-soliton solution splits into three different classes, depending on the sign of the parameter γ_1 and its difference from v_0 . The general form of the solutions agree with the trivial vacuum ones, but its behavior is modified by the presence of the v_0 parameter. The 1, 2 and 3-soliton solutions were explicit checked for equation (2.21). Moreover, the 1-soliton (4.13) with (4.12) and $m = 2$ was also verified to satisfy equation (5.1), using symbolic computational methods.

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Appendix. Matrix elements

Consider the vertex operator for $\hat{sl}(2)$,

$$F(\gamma, v_0) = \sum_{n=-\infty}^{\infty} (\gamma^2 - v_0^2)^{-n} \left[h^{(n)} + \frac{v_0 - \gamma}{2\gamma} \delta_{n,0} \hat{c} + E_{\alpha}^{(n)} (\gamma + v_0)^{-1} - E_{-\alpha}^{(n+1)} (\gamma - v_0)^{-1} \right]. \tag{A.1}$$

In the highest weight representation $\{|\lambda_0 \rangle, |\lambda_1 \rangle\}$ we have the following action of $\hat{sl}(2)$ operators:

$$\begin{aligned} E_{\alpha}^{(0)} |\lambda_a \rangle &= 0, \\ E_{\pm\alpha}^{(n)} |\lambda_a \rangle &= 0, \quad n > 0 \\ h^{(n)} |\lambda_a \rangle &= 0, \quad n > 0 \\ h^{(0)} |\lambda_a \rangle &= \delta_{a1} |\lambda_a \rangle \\ \hat{c} |\lambda_a \rangle &= |\lambda_a \rangle \end{aligned} \tag{A.2}$$

where $a = 0, 1$. Using the adjoint relations $(h^{(n)})^\dagger = h^{(-n)}$, $(E_\alpha^{(n)})^\dagger = E_{-\alpha}^{(-n)}$ and $\hat{c}^\dagger = \hat{c}$ we also know their actions on $\langle \lambda_a |$. From this, we have

$$\langle \lambda_0 | F(\gamma, v_0) | \lambda_0 \rangle = \frac{(v_0 - \gamma)}{2\gamma} \equiv c^-, \quad \langle \lambda_1 | F(\gamma, v_0) | \lambda_1 \rangle = \frac{(v_0 + \gamma)}{2\gamma} \equiv c^+. \quad (\text{A.3})$$

In order to calculate $\langle \lambda_a | F(\gamma_1, v_0) F(\gamma_2, v_0) | \lambda_a \rangle$, after distributing the products and keeping only non-trivial terms, we make use of the commutator rules to change the order. The double sum simplifies to a single sum, which can then be substituted for power series like $\sum_{n=0}^\infty x^n = 1/(1-x)$, $\sum_{n=1}^\infty x^n = x/(1-x)$ and $\sum_{n=1}^\infty nx^n = x/(1-x)^2$. The result is then given by

$$\begin{aligned} \langle \lambda_a | F(\gamma_1, v_0) F(\gamma_2, v_0) | \lambda_a \rangle &= \delta_{a1} + \frac{2(\gamma_1^2 - v_0^2)(\gamma_2^2 - v_0^2)}{(\gamma_1^2 - \gamma_2^2)^2} + \frac{v_0 - \gamma_1}{2\gamma_1} \delta_{a1} + \frac{v_0 - \gamma_2}{2\gamma_2} \delta_{a1} \\ &+ \frac{(\gamma_1 - v_0)(\gamma_2 - v_0)}{4\gamma_1\gamma_2} - \frac{(\gamma_1 - v_0)(\gamma_2 + v_0)}{\gamma_1^2 - \gamma_2^2} \delta_{a1} + \frac{(\gamma_1 + v_0)(\gamma_2 - v_0)}{\gamma_1^2 - \gamma_2^2} \delta_{a1} \\ &- (\gamma_1 - v_0)(\gamma_2 + v_0) \frac{\gamma_2^2 - v_0^2}{(\gamma_1^2 - \gamma_2^2)^2} - (\gamma_1 + v_0)(\gamma_2 - v_0) \frac{\gamma_1^2 - v_0^2}{(\gamma_1^2 - \gamma_2^2)^2}. \end{aligned} \quad (\text{A.4})$$

This expression can be further simplified to

$$\begin{aligned} \langle \lambda_0 | F(\gamma_1, v_0) F(\gamma_2, v_0) | \lambda_0 \rangle &= \frac{(\gamma_1 - v_0)(\gamma_2 - v_0)}{4\gamma_1\gamma_2} \left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \right)^2 = c_1^- c_2^- a_{12}, \\ \langle \lambda_1 | F(\gamma_1, v_0) F(\gamma_2, v_0) | \lambda_1 \rangle &= \frac{(\gamma_1 + v_0)(\gamma_2 + v_0)}{4\gamma_1\gamma_2} \left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \right)^2 = c_1^+ c_2^+ a_{12}, \end{aligned} \quad (\text{A.5})$$

where $c_i^\pm = c^\pm(\gamma_i)$, see (A.3), and we have defined

$$a_{ij} = \left(\frac{\gamma_i - \gamma_j}{\gamma_i + \gamma_j} \right)^2. \quad (\text{A.6})$$

Note that (A.5) $\rightarrow 0$ when $\gamma_2 \rightarrow \gamma_1$. This proves the nilpotency property of the vertex operator when evaluated within diagonal states $|\lambda_0\rangle$ and $|\lambda_1\rangle$.

A more tedious calculation shows that

$$\begin{aligned} \langle \lambda_0 | F(\gamma_1, v_0) F(\gamma_2, v_0) F(\gamma_3, v_0) | \lambda_0 \rangle &= c_1^- c_2^- c_3^- a_{12} a_{13} a_{23}, \\ \langle \lambda_1 | F(\gamma_1, v_0) F(\gamma_2, v_0) F(\gamma_3, v_0) | \lambda_1 \rangle &= c_1^+ c_2^+ c_3^+ a_{12} a_{13} a_{23}. \end{aligned} \quad (\text{A.7})$$

In general, using Wick theorem, it is possible to show that

$$\begin{aligned} \langle \lambda_0 | \prod_{i=1}^n F(\gamma_i, v_0) | \lambda_0 \rangle &= \prod_{i=1}^n c_i^- \prod_{i,j=1, i<j}^n a_{ij}, \\ \langle \lambda_1 | \prod_{i=1}^n F(\gamma_i, v_0) | \lambda_1 \rangle &= \prod_{i=1}^n c_i^+ \prod_{i,j=1, i<j}^n a_{ij}. \end{aligned} \quad (\text{A.8})$$

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